Clifford algebra analogue of the Cartan theorem for symmetric pairs

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arXiv:2504.20917,2411.11663

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The Hopf–Koszul–Samelson theorem

Let G be a compact connected simple Lie group and $\mathfrak{g} = \operatorname{Lie} G$.

$$H(G) = H(\mathfrak{g}) = (\textstyle\bigwedge \mathfrak{g})^{\mathfrak{g}} = \bigwedge P_\wedge(\mathfrak{g}),$$

where $P_\wedge(\mathfrak{g})$ is the subspace of primitive invariants: $\bigwedge(\mathfrak{g})^{\mathfrak{g}} = \mathbb{C}1 \oplus P_\wedge(\mathfrak{g}) \oplus ((\bigwedge^+ \mathfrak{g})^{\mathfrak{g}})^2$.
The space of primitive invariants has a natural $\mathbb{Z}$-grading $P_\wedge(\mathfrak{g}) = \bigoplus_{i=1}^{\operatorname{rank}\mathfrak{g}} P_\wedge^{2m_i+1}(\mathfrak{g})$, where m_i are exponents of $\mathfrak{g}$. For example, $P_\wedge(\mathfrak{sl}_n) = P_\wedge^3 \oplus P_\wedge^5 \oplus \ldots \oplus P_\wedge^{2n-1}$.

Clifford algebra analogue of the Hopf–Koszul–Samelson theorem

Since $\mathfrak{g}$ is simple, it admits a non-degenerate invariant symmetric bilinear form B. Recall that

$$\mathrm{Cl}(\mathfrak{g}) = T(\mathfrak{g})/\langle x \otimes y + y \otimes x - 2B(x, y)\rangle$$

is a filtered deformation of $\bigwedge(\mathfrak{g})$. The Chevalley (or quantisation) map is given by

$$q\colon \textstyle\bigwedge(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g}) \qquad z_{i_1} \wedge \ldots \wedge z_{i_k} \mapsto z_{i_1} \cdot \ldots \cdot z_{i_k},$$

where z_i is an orthonormal basis of $\mathfrak{g}$ and $i_1 < i_2 < \ldots < i_k$.
The Chevalley map is $\mathfrak{k}$-equivariant and compatible with the filtration on $\mathrm{Cl}(\mathfrak{g})$:

$$\mathrm{Cl}^{(k)}(\mathfrak{g}) := q\left(\bigoplus_{m=0}^{k} \textstyle\bigwedge^m(\mathfrak{g})\right) \qquad \text{for } k = 0, \ldots, \dim(\mathfrak{g}).$$

Theorem (Kostant [5])

There is a filtered algebra isomorphism $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}} = \mathrm{Cl}(q(P_\wedge(\mathfrak{g})), \tilde{B})$.

Symmetric pairs

A subalgebra $\mathfrak{k}$ is called *symmetric* if there is an involution θ of $\mathfrak{g}$ such that $\mathfrak{k} = \mathfrak{g}^\theta$. We have a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{k} := \{x \in \mathfrak{g} \mid \theta(x) = x\} \qquad \mathfrak{p} := \{x \in \mathfrak{g} \mid \theta(x) = -x\}.$$

(Assume that it is an orthogonal decomposition with respect to B.)
Let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$ and $\mathfrak{a} \subseteq \mathfrak{p}$. Such a Cartan subalgebra is called fundamental, or maximally compact.

Symmetric pairs: examples

G	K	$\mathfrak{g}$	$\mathfrak{k}$	$\dim \mathfrak{a}$
$SU(2n)$	$SO(2n)$	$\mathfrak{sl}(2n)$	$\mathfrak{o}(2n)$	$n-1$
$SU(2n+1)$	$SO(2n+1)$	$\mathfrak{sl}(2n+1)$	$\mathfrak{o}(2n+1)$	n
$SU(p+q)$	$S(U(p) \times U(q))$	$\mathfrak{sl}(p+q)$	$\mathfrak{gl}(p) \oplus \mathfrak{sl}(q)$	0
$SO(2p+1, 2q+1)$	$SO(2p+1) \times SO(2q+1)$	$\mathfrak{o}(2p+2q+2)$	$\mathfrak{o}(2p+1) \oplus \mathfrak{o}(2q+1)$	1

Spin module(s) over $\mathrm{Cl}(\mathfrak{p})$ as $\mathfrak{k}$-modules

Let $S_{\mathfrak{p}}^{\pm}$ denote irreducible modules for $\mathrm{Cl}(\mathfrak{p})$. The quantum moment map $\alpha\colon U(\mathfrak{k}) \to \mathrm{Cl}(\mathfrak{p})$ is defined by

$$\alpha\colon \mathfrak{k} \xrightarrow{\mathrm{ad}} \mathfrak{o}(\mathfrak{p}) \xrightarrow{\cong} \textstyle\bigwedge^2(\mathfrak{p}) \xrightarrow{q} \mathrm{Cl}^{(2)}(\mathfrak{p}).$$

We have two (not necessary restricted) root systems $\Delta(\mathfrak{g}, \mathfrak{t})$ and $\Delta(\mathfrak{k}, \mathfrak{t})$.
Consider the corresponding Weyl groups $W_{\mathfrak{g},\mathfrak{t}}$ and $W_{\mathfrak{k},\mathfrak{t}}$. Let $W^1 = W_{\mathfrak{g},\mathfrak{t}}/W_{\mathfrak{k},\mathfrak{t}}$.
Let $\mathfrak{a}^+$ and $\mathfrak{a}^-$ be maximal isotropic subspaces orthogonal to each other, then as $\mathfrak{k}$-modules (here the $\mathfrak{k}$-action on $\bigwedge \mathfrak{a}^+$ is trivial),

$$S_{\mathfrak{p}}^+ = \sum_{w \in W^1} \textstyle\bigwedge \mathfrak{a}^+ \otimes V_{w\rho_{\mathfrak{g}} - \rho_{\mathfrak{k}}}, \qquad S_{\mathfrak{p}}^- = \sum_{w \in W^1} \bigwedge \mathfrak{a}^- \otimes V_{w\rho_{\mathfrak{g}} - \rho_{\mathfrak{k}}},$$

Algebra of projections

Define an algebra $\mathrm{Pr}(S_{\mathfrak{p}})$ with a basis p_w, $w \in W^1$ such that $p_{w_i} p_{w_j} = \delta_{w_i, w_j} p_{w_i}$.

Proposition (Calvert–Grizelj–Krutov–Pandžić)

There is an algebra isomorphism $\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}} \cong \mathrm{Cl}(\mathfrak{a}) \otimes \mathrm{Pr}(S_{\mathfrak{p}})$. *(Note that it is not a filtered algebra isomorphism!) Under the isomorphism* $Z(U(\mathfrak{k})) \cong S(\mathfrak{t})^{W_{\mathfrak{k},\mathfrak{t}}}$*, the restriction of* α *to* $S(\mathfrak{t})^{W_{\mathfrak{k},\mathfrak{t}}}$ *is the sum of evaluations at* $w\rho_{\mathfrak{g}}$ *for* $w \in W^1$*:*

$$\alpha(s) = \sum_{w \in W^1} s(w\rho_{\mathfrak{g}}) p_w, \qquad s \in S(\mathfrak{t})^{W_{\mathfrak{k},\mathfrak{t}}}.$$

Relative transgression theorem

Consider the relative Weil algebra $W(\mathfrak{g}, K) = (S(\mathfrak{g}^*) \otimes \bigwedge \mathfrak{p}^*)^K$ with the differential given by $\mathrm{d}_W = \mathrm{d}_{\mathrm{CE}} + \mathrm{d}_{\mathrm{K}}$ (the sum of Chevalley–Eilenberg and Koszul differentials).
Consider two projections: $\pi_{\wedge\mathfrak{p}^*}\colon W(\mathfrak{g}, K) \to (\wedge \mathfrak{p}^*)^K$ and $\mathrm{pr}_{\mathfrak{t}}\colon S(\mathfrak{g}^*)^{\mathfrak{g}} \to S(\mathfrak{k}^*)^K$.
We have that $H(W(\mathfrak{g}, K), \mathrm{d}_W) \cong S(\mathfrak{g}^*)^K$. Set $T(\mathfrak{g}, K) = \{p \in S(\mathfrak{g}^*)^{\mathfrak{g}} \mid \mathrm{pr}_{\mathfrak{k}}(p) = 0\}$.

For $p \in T(\mathfrak{g}, K)$ there is a $C_p \in W(\mathfrak{g}, K)$ such that $\mathrm{d}_W(C_p) = p$. We define the transgression map $\mathsf{t}_{\wedge\mathfrak{p}^*}\colon T(\mathfrak{g}, K) \to (\bigwedge \mathfrak{p}^*)^K$ by $\mathsf{t}_{\wedge\mathfrak{p}^*}(p) = \pi_{\wedge\mathfrak{p}^*}(C_p)$.

Theorem (Calvert–Grizelj–Krutov–Pandžić)

Let $P_\wedge(\mathfrak{p}) = \operatorname{Res}_{\mathfrak{p}}^{\mathfrak{g}} P_\wedge(\mathfrak{g})$. *Let* $(\mathfrak{g}, \mathfrak{k})$ *be one of the following pairs:*

$$(\mathfrak{sl}(n), \mathfrak{o}(n)), \qquad (\mathfrak{sl}(2n), \mathfrak{sp}(2n)), \qquad (\mathfrak{e}(6), \mathfrak{f}(4)),$$

then $\dim P_\wedge(\mathfrak{p}) = \dim \mathfrak{a}$ *and the map* $\mathsf{t}_{\wedge\mathfrak{p}^*}$ *satisfies*

$$\ker \mathsf{t}_{\wedge\mathfrak{p}^*} = S^+(\mathfrak{g}^*)^{\mathfrak{g}} \cdot T(\mathfrak{g}, K), \qquad \operatorname{im} \mathsf{t}_{\wedge\mathfrak{p}^*} = P_\wedge(\mathfrak{p}),$$

and $\mathsf{t}_{\wedge\mathfrak{p}^*}\colon T(\mathfrak{g}, K)/S^+(\mathfrak{g}^*)^{\mathfrak{g}} \cdot T(\mathfrak{g}, K) \longrightarrow P_\wedge(\mathfrak{p})$ *is an isomorphism of vector spaces.*

This generalises the results for $\wedge\mathfrak{g}^*$ of Chevalley [3] and H. Cartan [2].

Clifford algebra analogue of Cartan's theorem

Since B is non-degenerate on $\mathfrak{g}$ and $\mathfrak{p}$, we have that $\bigwedge \mathfrak{g}^* \cong \bigwedge \mathfrak{g}$ and $\bigwedge \mathfrak{p}^* \cong \bigwedge \mathfrak{p}$.

Theorem (Calvert–Grizelj–Krutov–Pandžić)

Let (G, K) *be a compact symmetric pair such that* G *is simple and connected. Assume that* $(\mathfrak{g}, \mathfrak{k})$ *is different from* $(\mathfrak{e}(6), \mathfrak{sp}(8))$.
(a) With the above notation, the inclusion $q(P_\wedge(\mathfrak{p})) \hookrightarrow \mathrm{Cl}(\mathfrak{p})^K$ *extends to a filtered algebra homormorpism*

$$\mathrm{Cl}(q(P_\wedge(\mathfrak{p})), \tilde{B}) \to \mathrm{Cl}(\mathfrak{p})^K,$$

which is an isomorphism in the primary cases, i.e., when the spin module $S_{\mathfrak{p}}$ *contains only one* $\mathfrak{k}$*-type. For the “almost primary cases”* $(G, K) = (SU(2n), SO(2n))$*, the statement remains true if* K *is replaced by* $O(2n)$.
(b) There is an algebra isomorphism

$$\mathrm{Cl}(q(P_\wedge(\mathfrak{p})), \tilde{B}) \otimes \mathrm{Pr}(S_{\mathfrak{p}}) \cong \mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}}.$$

(c) There is a filtered algebra isomorphism

$$\mathrm{Pr}(S_{\mathfrak{p}}) \cong \mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k},\mathfrak{t}}}/I_\rho,$$

where I_ρ *is the ideal of* $\mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k},\mathfrak{t}}}$ *generated by* $a \in \mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{g},\mathfrak{t}}}$ *such that* $a(\rho) = 0$.
(d) Passing to the associated graded algebras we recover the well known Cartan (or Cartan-Borel) Theorem for symmetric spaces

$$\textstyle\bigwedge P_\wedge(\mathfrak{p}) \otimes \mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k},\mathfrak{t}}}/I_+ \cong (\bigwedge \mathfrak{p})^{\mathfrak{k}},$$

where $I_+ = \operatorname{gr} I_\rho$ *is the ideal of* $\mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k},\mathfrak{t}}}$ *generated by elements of* $\mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{g},\mathfrak{t}}}$ *with zero constant term.*

Harish–Chandra projections

Fix a choice $\Delta^+(\mathfrak{k}, \mathfrak{t})$ of positive $\mathfrak{t}$-roots in $\mathfrak{k}$. Each $w \in W^1$ fixes the choice $\Delta_w^+(\mathfrak{g}, \mathfrak{t})$ of positive $\mathfrak{t}$-root in $\mathfrak{g}$. Consider the corresponding triangular decomposition $\mathfrak{g} = \mathfrak{n}_w^+ \oplus \mathfrak{h} \oplus \mathfrak{n}_w^-$. We have a decomposition

$$\mathrm{Cl}(\mathfrak{p}) = \mathrm{Cl}(\mathfrak{a}) \oplus \big(\mathrm{Cl}(\mathfrak{p} \cap \mathfrak{n}_w^+)\, \mathrm{Cl}(\mathfrak{p}) + \mathrm{Cl}(\mathfrak{p})\, \mathrm{Cl}(\mathfrak{p} \cap \mathfrak{n}_w^-)\big)$$

Proposition (Calvert–Grizelj–Krutov–Pandžić)

For each $w \in W^1$ *the corresponding Harish–Chandra projection*

$$\mathrm{hc}_w\colon \mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}} \to \mathrm{Cl}(\mathfrak{a})$$

is a surjective algebra homomorphism.

We have that $\mathrm{hc}_w(q(P_\wedge(\mathfrak{p}))) = \mathfrak{a} \subset \mathrm{Cl}^{(1)}(\mathfrak{a})$. Let $\check{\mathfrak{g}}$ be the Lie algebra defined by the dual root system and $(\check{e}, \check{h}, \check{f})$ be the principal $\mathfrak{sl}_2$-triple in $\check{\mathfrak{g}}$. Note $\check{\mathfrak{h}}^* \cong \mathfrak{h}$. Note that $\mathfrak{a} \subset \mathfrak{h} \cong \check{\mathfrak{h}}^*$. Define two filtrations on $\mathfrak{a}$ by

$$F^{(m)}\mathfrak{a} = \bigoplus_{i=0}^{m} \mathrm{hc}_w(q(P_\wedge^i(\mathfrak{p}))), \qquad \mathcal{F}^{(m)}\mathfrak{a} = \{h \in \mathfrak{a} \mid (\mathrm{ad}_{\check{e}}^*)^{m+1} h = 0\}.$$

Theorem (Calvert–Grizelj–Krutov–Pandžić)

For all $m \in \mathbb{N}$*, the filtration* $F^{(m)}\mathfrak{a}$ *doesn't depend on* w *and* $\mathcal{F}^{(m)}\mathfrak{a} = F^{(m)}\mathfrak{a}$.

For $\mathrm{Cl}(\mathfrak{g})$ this was proved by Joseph [4] and Aleekseev–Moreau [1].

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