Clifford algebra analogue of the Cartan theorem for symmetric pairs

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I. Introduction and the main result

Hopf–Koszul–Samelson theorem

Let G be a compact connected simple Lie group and $\mathfrak{g} = \mathrm{Lie}\, G$.

$$H(G) = H(\mathfrak{g}) = (\bigwedge \mathfrak{g})^{\mathfrak{g}} = \bigwedge P_{\wedge}(\mathfrak{g}),$$

where $P_{\wedge}(\mathfrak{g})$ is the subspace of primitive invariants.

$$(\bigwedge \mathfrak{g})^{\mathfrak{g}} = \mathbb{C}1 \oplus P_{\wedge}(\mathfrak{g}) \oplus I_{+}^{2}, \qquad B_{\wedge}(\phi, \psi) = (\iota_{\phi^T} \psi)_{[0]}.$$

where $I_{+} = (\bigwedge^{+} \mathfrak{g})^{\mathfrak{g}}$ is the augmentation ideal.
Equivalently

$$P_{\wedge}(\mathfrak{g}) = \{p \in (\bigwedge \mathfrak{g})^{\mathfrak{g}} \mid \Delta p = 1 \otimes p + p \otimes 1\}.$$

The space of primitive invariants has a natural $\mathbb{Z}$-grading

$$P_{\wedge}(\mathfrak{g}) = \bigoplus_{i=1}^{\mathrm{rank}\mathfrak{g}} P_{\wedge}^{2m_i+1}(\mathfrak{g}),$$

where m_i are exponents of $\mathfrak{g}$. $P_{\wedge}(\mathfrak{sl}_n) = P_{\wedge}^{3} \oplus P_{\wedge}^{5} \oplus \ldots \oplus P_{\wedge}^{2n-1}$.

Clifford analogue Hopf–Koszul–Samelson theorem

- Let $\mathfrak{g}$ be a Lie algebra with NIS $B\colon \mathfrak{g}\otimes\mathfrak{g}\to\mathbb{C}$.

$$\mathrm{Cl}(\mathfrak{g}) = T(\mathfrak{g})/\langle x\otimes y + y\otimes x - 2B(x,y)\rangle$$

is a filtered deformation of $\bigwedge(\mathfrak{g})$.

The Chevalley (or quantisation) map

$$q\colon \textstyle\bigwedge\mathfrak{g}\to \mathrm{Cl}(\mathfrak{g}) \qquad z_{i_1}\wedge\ldots\wedge z_{i_k}\mapsto z_{i_1}\cdot\ldots\cdot z_{i_k},$$

where z_i is an orthonormal basis of $\mathfrak{g}$ and $i_1 < i_2 < ... < i_k$.

- q is $\mathfrak{g}$-equivariant and compatible with the standard filtration

$$\mathrm{Cl}^{(k)}(\mathfrak{g}) = q\left(\bigoplus_{m=0}^{k}\textstyle\bigwedge^m\mathfrak{g}\right) \qquad \text{for } k=0,\ldots,\dim(\mathfrak{g}).$$

Theorem (Kostant'1999)

The inclusion $q(P_\wedge(\mathfrak{g}))\hookrightarrow \mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$ extends to an isomorphism of filtered superalgebras

$$\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}} = \mathrm{Cl}(q(P_\wedge(\mathfrak{g})),\tilde{B}), \quad \tilde{B}(\phi,\psi) = (\iota_\phi\psi)_{[0]}.$$

Symmetric pairs

• A subalgebra $\mathfrak{k}$ is called *symmetric* if there is an involution θ of $\mathfrak{g}$ such that $\mathfrak{k} = \mathfrak{g}^\theta$.

• We have a decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$
$$\mathfrak{k} := \{x \in \mathfrak{g} \mid \theta(x) = x\} \qquad \mathfrak{p} := \{x \in \mathfrak{g} \mid \theta(x) = -x\}$$

(Assume that it is orthogonal decomposition with respect to B.)

• Let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$ and $\mathfrak{a} \subseteq \mathfrak{p}$. ($\mathfrak{h}$ is called a fundamental Cartan subalgebra.)

$$\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$$

• typical examples come from real reductive groups and symmetric spaces.

Cartan's theorem for symmetric pairs

• Hopf–Koszul–Samelson

$$(\bigwedge \mathfrak{g})^{\mathfrak{g}} = \bigwedge P_{\wedge}(\mathfrak{g}),$$

where $P_{\wedge}(\mathfrak{g}) \subset (\bigwedge \mathfrak{g})^{\mathfrak{g}}$ is the subspace of primitive invariants
• Cartan's theorem: there is an isomorphism of graded algebras

$$(\bigwedge \mathfrak{p})^{\mathfrak{k}} \cong \bigwedge P_{\mathrm{Sam}}(\mathfrak{g}, \mathfrak{k}) \otimes M,$$

where
– M is a commutative algebra generated in even degrees (characteristic subalgebra);
– the Samelson subspace of $(\mathfrak{g}, \mathfrak{k})$ if given by

$$P_{\mathrm{Sam}}(\mathfrak{g}, \mathfrak{k}) = \{p \in P_{\wedge}(\mathfrak{g}) \mid \theta(p) = -p\} \subset (\bigwedge \mathfrak{g})^{\mathfrak{g}}.$$

• $\mathfrak{k}$-equivariant cohomologies of $\bigwedge(\mathfrak{g})$.
• $H(G/K) \cong (\bigwedge \mathfrak{p})^{\mathfrak{k}}$.

Spin module(s) $S_{\mathfrak{p}}$

Let $S_{\mathfrak{p}}$ denotes an irreducible module for $\mathrm{Cl}(\mathfrak{p})$.
We have two (not necessary restricted) root systems

$$\Delta(\mathfrak{g},\mathfrak{t}) \quad \text{and} \quad \Delta(\mathfrak{k},\mathfrak{t}).$$

Consider the corresponding Weyl groups $W_{\mathfrak{g},\mathfrak{t}}$ and $W_{\mathfrak{k},\mathfrak{t}}$. Set

$$W^1 = W_{\mathfrak{g},\mathfrak{t}}/W_{\mathfrak{k},\mathfrak{t}}$$

(minimal length representatives).
As $\mathfrak{k}$-modules,

$$S_{\mathfrak{p}} = \bigoplus_{w\in W^1} \bigwedge \mathfrak{a}^+ \otimes V_{w\rho_{\mathfrak{g}}-\rho_{\mathfrak{k}}},$$

where the $\mathfrak{k}$-action on $\bigwedge \mathfrak{a}^+$ is trivial.

The structure of $\mathfrak{k}$-invariants in $\mathrm{Cl}(\mathfrak{p})$

Define an algebra $\mathrm{Pr}(S_{\mathfrak{p}})$ isomorphic to $\mathbb{C}^{|W^1|}$ with basis of idempotents labeled by p_w, $w \in W^1$ such that

$$p_{w_i} p_{w_j} = \delta_{w_i, w_j} p_{w_i}.$$

Lemma

The $\mathfrak{k}$-invariants in $\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}}$ are isomorphic as an algebra to

$$\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}} \cong \mathrm{Cl}(\mathfrak{a}) \otimes \mathrm{Pr}(S_{\mathfrak{p}}).$$

*(Note that it is **not** a filtered algebra isomorphism!)*

Note that

$$\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}} = (\mathrm{End}(S_{\mathfrak{p}}))^{\mathfrak{k}} = \mathrm{End}_{\mathfrak{k}}(S_{\mathfrak{p}}).$$

Apply to

$$S_{\mathfrak{p}} = \sum_{w \in W^1} \bigwedge \mathfrak{a}^+ \otimes V_{w\rho_{\mathfrak{g}} - \rho_{\mathfrak{k}}},$$

Difficulty. Kostant used the Hodge theory to describe $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$. This is not available for $\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}}$ since the differential is zero.

We are interested in the following cases $(\mathfrak{g}, \mathfrak{k})$:

- $S_{\mathfrak{p}}$ has only one $\mathfrak{k}$-type (*primary cases*):

$$(\mathfrak{sl}(2n+1), \mathfrak{o}(2n+1)), \qquad (\mathfrak{sl}(2n), \mathfrak{sp}(2n)), \qquad (\mathfrak{e}(6), \mathfrak{f}(4)).$$

- $S_{\mathfrak{p}}$ has only two $\mathfrak{k}$-type and $\dim \mathfrak{a} > 1$ (*almost primary cases*):

$$(\mathfrak{sl}(2n), \mathfrak{o}(2n)).$$

Main theorem

Let (G, K) be a compact symmetric pair such that G is simple and connected. Assume that $(\mathfrak{g}, \mathfrak{k})$ is different from $(\mathfrak{e}(6), \mathfrak{sp}(8))$.

Set $P_{\mathrm{Cl}}(\mathfrak{p}) := q(\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}}(P_{\wedge}(\mathfrak{g})))$.

Theorem (Calvert–Grizelj–K.–Pandžić)

(a) With the above notation, the inclusion $P_{\mathrm{Cl}}(\mathfrak{p}) \hookrightarrow \mathrm{Cl}(\mathfrak{p})^K$ extends to a filtered algebra homormorpism

$$\mathrm{Cl}(P_{\mathrm{Cl}}(\mathfrak{p}), \tilde{B}) \to \mathrm{Cl}(\mathfrak{p})^K,$$

which is an isomorphism when the spin module S contains only one $\mathfrak{k}$-type. For the $(G, K) = (SU(2n), SO(2n))$, the statement remains true if K is replaced by $O(2n)$.
(b) There is an algebra isomorphism

$$\mathrm{Cl}(P_{\mathrm{Cl}}(\mathfrak{p}), \tilde{B}) \otimes \mathrm{Pr}(S) \cong \mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}}.$$

II. Weil aglebras and transgression theorems

G-differential algebras I

- $\mathfrak{g}$ be a Lie algebra,
- $\bigwedge[\xi]$ is the Grassmann algebra generated by ξ,
- $\mathrm{d} = \partial_\xi$ is the differential on $\bigwedge[\xi]$.

Consider a semisimple $\mathbb{Z}$-graded Lie superalgebra

$$\widehat{\mathfrak{g}} = \bigwedge[\xi] \otimes \mathfrak{g} \oplus \mathbb{C}\mathrm{d} = \widehat{\mathfrak{g}}_{-1} \oplus \widehat{\mathfrak{g}}_0 \oplus \widehat{\mathfrak{g}}_1,$$

where

$$\begin{aligned} \widehat{\mathfrak{g}}_{-1} &= \mathrm{Span}(\iota_x \mid x \in \mathfrak{g}), && \text{contractions} \\ \widehat{\mathfrak{g}}_0 &= \mathrm{Span}(L_x \mid x \in \mathfrak{g}), && \text{Lie derivatives} \\ \widehat{\mathfrak{g}}_1 &= \mathrm{Span}(\mathrm{d}). \end{aligned}$$

The explicit brackets in $\widehat{\mathfrak{g}}$ are as follows

$$\begin{aligned} [L_x, L_y] &= L_{[x,y]}, & [L_x, \iota_y] &= \iota_{[x,y]}, & [\mathrm{d}, \iota_x] &= L_x, \\ [\mathrm{d}, \mathrm{d}] &= 0, & [L_x, \mathrm{d}] &= 0, & [\iota_x, \iota_y] &= 0, \end{aligned}$$

for all $x, y \in \mathfrak{g}$.

- A $\mathfrak{g}$-*differential space* is a super vector space V, together with a structure of $\widehat{\mathfrak{g}}$-representation $\rho\colon \widehat{\mathfrak{g}} \to \mathfrak{gl}(V)$.

- A $\mathfrak{g}$-*differential algebra* is a super algebra $\mathcal{A}$, together with a structure of a $\mathfrak{g}$-differential space, such that $\rho(x) \in \mathrm{Der}(A)$ for all $x \in \widehat{\mathfrak{g}}$.

- G be a compact Lie group and denote by $\mathfrak{g}$ its Lie algebra. A G-*differential algebra* is a superalgebra $\mathcal{A}$ with a representation ρ of G by automorphisms of $\mathcal{A}$ and a structure of $\mathfrak{g}$-differential algebra on $\mathcal{A}$ which satisfies

$$\frac{d}{dt}\rho(\exp(-tx))\Big|_{t=0} = L_x, \qquad \rho(g)L_x\rho(g^{-1}) = L_{\mathrm{Ad}_g x},$$
$$\rho(g)\iota_x\rho(g^{-1}) = \iota_{\mathrm{Ad}_g x}, \qquad \rho(g)\mathrm{d}\rho(g^{-1}) = \mathrm{d},$$

for all $g \in G$ and $x \in \mathfrak{g}$, where Ad denotes the adjoint action of G on $\mathfrak{g}$.

Let $\mathcal{A}$ be a G-differential algebra.

$$\mathcal{A}_{G-\mathrm{bas}} = \mathcal{A}^G \cap \{a \in \mathcal{A} \mid \iota_x a = 0, \forall x \in \mathfrak{g}\}.$$

Example. $\mathcal{A} = \bigwedge \mathfrak{g}^*$ is a G-differential algebra with

$$L_x = \mathrm{ad}_x^*, \quad \iota_x(f) = \langle f, x \rangle, \quad \mathrm{d}_\wedge = \frac{1}{2} \sum_a f_a \circ L_{e_a}.$$

If G/K is a compact symmetric space then

$$(\bigwedge \mathfrak{g}^*)_{K-\mathrm{bas}} = (\bigwedge \mathfrak{p}^*)^K$$

Note that $H(G/K) \cong (\bigwedge \mathfrak{p}^*)^K$ and the restriction of the differential $\mathrm{d}_\wedge$ of $(\bigwedge \mathfrak{p}^*)^K$ is zero.

The α-map

Let V be a $\mathfrak{l}$-module with nondegenerate symmetric invariant bilinear form B_V.

- Set

$$\alpha_V : \mathfrak{l} \xrightarrow{\mathrm{ad}} \mathfrak{o}(V, B_V) \xrightarrow{\simeq} \bigwedge^2 V \xrightarrow{q} \mathrm{Cl}^{(2)}(V),$$

- Extend (by the universal property) to quantum moment map

$$\alpha : U(\mathfrak{l}) \longrightarrow \mathrm{Cl}(V).$$

Note that the $\mathfrak{l}$-action on $\mathrm{Cl}(V)$ is (quantum in sense of Lu) Hamiltonian

$$L_x = [\alpha_V(x),\ -] \qquad \text{for } x \in \mathfrak{l}.$$

Example. Let $\mathcal{A} = \mathrm{Cl}(\mathfrak{g})$ then

$$L_x = [\alpha_{\mathfrak{g}}(x), -], \qquad \iota_x = [\frac{1}{2}x, -], \qquad \mathrm{d}_{\mathrm{Cl}} = [q(\phi), -],$$

where

$$\phi = \frac{1}{24} \sum_{i,j} [e_i, e_j] \wedge f_j \wedge f_i \in (\bigwedge \mathfrak{g})^{\mathfrak{g}}$$

is the Cartan 3-tensor.

Note that $q \colon \bigwedge \mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ intertwines L_x and ι_x but not differentials.

For G/K we have that

$$\mathrm{Cl}(\mathfrak{g})_{K-\mathrm{bas}} = \mathrm{Cl}(\mathfrak{p})^K.$$

$$\mathrm{gr}\, \mathrm{Cl}(\mathfrak{p})^K \cong (\bigwedge \mathfrak{p}^*)^K.$$

Let

$$W(\mathfrak{g}) = S(\mathfrak{g}^*) \otimes \wedge\mathfrak{g}^*$$

with generators $\widehat{f_i}$ of $S(\mathfrak{g}^*) \otimes 1$ and f_i of $1 \otimes \bigwedge\mathfrak{g}^*$.
For $x \in \mathfrak{g}$ set

$$L_x = \mathrm{ad}_x^* \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_x^*, \qquad \iota_x = \mathrm{id} \otimes \iota_x.$$

Define the differential

$$\mathrm{d}_W = \underbrace{\sum_a L_{e_a} \otimes f_a + \mathrm{id} \otimes \mathrm{d}_\wedge}_{=\mathrm{d}_{\mathrm{CE}}} + 2 \underbrace{\sum_a \widehat{f}_a \otimes \iota_{e_a}}_{=\mathrm{d}_K}.$$

Then $W(\mathfrak{g})$ is a G-differential algebra.

Note that $S^+(\mathfrak{g})^{\mathfrak{g}} \otimes 1 \subset \ker \mathrm{d}_W$.

The transgression map in $W(\mathfrak{g})$

Define the projection

$$\pi_{\wedge\mathfrak{g}^*} : W(\mathfrak{g}) \to \bigwedge \mathfrak{g}^*$$

along the augmentation ideal $S^+(\mathfrak{g}^*) \otimes \bigwedge \mathfrak{g}^*$.

The algebras $W(\mathfrak{g})$ and $W(\mathfrak{g})^{\mathfrak{g}}$ are acyclic

$$H(W(\mathfrak{g}), \mathrm{d}_W) = H(W(\mathfrak{g})^{\mathfrak{p}}, \mathrm{d}_W) = \mathbb{C}.$$

For $p \in S^+(\mathfrak{g})^{\mathfrak{g}}$ we have that $\mathrm{d}_W p = 0$. Hence there is a $C_p \in W(\mathfrak{g})^{\mathfrak{g}}$ such that $\mathrm{d}_W C_p = p$. The element C_p is called a cochain of transgression.

For $p \in S^+(\mathfrak{g})^{\mathfrak{g}}$ the *transgression* of p is defined

$$\mathsf{t}_{\wedge\mathfrak{g}^*}(p) = \pi_{\wedge\mathfrak{g}^*}(C_p)$$

We define the *transgression map*

$$\mathsf{t}_{\wedge\mathfrak{g}^*} : S^+(\mathfrak{g})^{\mathfrak{g}} \to (\bigwedge \mathfrak{g}^*)^{\mathfrak{g}}.$$

Transgression theorem

Let $\mathfrak{g}$ be a complex reductive Lie algebra.
Let $P_S(\mathfrak{g})$ be an (orthogonal) compliment to $(S^+(\mathfrak{g})^\mathfrak{g})^2$:

$$S^+(\mathfrak{g})^\mathfrak{g} = P_S(\mathfrak{g}) \oplus (S^+(\mathfrak{g})^\mathfrak{g})^2$$

Theorem (Chevalley)

The transgression map $\mathtt{t}_{\wedge\mathfrak{g}^*}$ *satisfies*

$$\ker \mathtt{t}_{\wedge\mathfrak{g}^*} = (S^+(\mathfrak{g})^\mathfrak{g})^2, \qquad \operatorname{im} \mathtt{t}_{\wedge\mathfrak{g}^*} = P_\wedge(\mathfrak{g}).$$

In other words we have an isomorphism of vector spaces

$$\mathtt{t}_{\wedge\mathfrak{g}^*} : P_S(\mathfrak{g}) \to P_\wedge(\mathfrak{g}).$$

Recall that $\dim P_\wedge(\mathfrak{g}) = \dim \mathfrak{h}$.

Define the quantum Weil algebra as

$$\mathcal{W}(\mathfrak{g}) = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g}).$$

Consider the cubic Dirac element

$$D = \sum \widehat{e}_a \otimes f_a + 1 \otimes q(\phi),$$

where ϕ is the Cartan 3-tensor.
Set

$$L_x = \mathrm{ad}_x \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_x, \quad \iota_x = \mathrm{id} \otimes \iota_x, \quad \mathrm{d}_{\mathcal{W}} = [D, -]$$

then $\mathcal{W}(\mathfrak{g})$ is a G-differential algebra.

The transgression map in $\mathcal{W}(\mathfrak{g})$ I

There is a isomorphism of G-differential space

$$q_W : W(\mathfrak{g}) \to \mathcal{W}(\mathfrak{g}),$$

which restricts to the Chevalley map on $\bigwedge \mathfrak{g}$ and the Duflo map on $S(\mathfrak{g})$.
Hence we get that

$$H(\mathcal{W}(\mathfrak{g}), \mathrm{d}_{\mathcal{W}}) = H(\mathcal{W}(\mathfrak{g})^{\mathfrak{g}}, \mathrm{d}_{\mathcal{W}}) = \mathbb{C}.$$

Define the projection

$$\pi_{\mathrm{Cl}(\mathfrak{g})} : \mathcal{W}(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g})$$

along $U^+(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g})$.
We have that

$$q \circ \pi_{\wedge \mathfrak{g}} = \pi_{\mathrm{Cl}(\mathfrak{g})} \circ q_W.$$

The transgression map in $\mathcal{W}(\mathfrak{g})$ II

Since $\mathcal{W}(\mathfrak{g})$ is acyclic we can define the transgression map.

For $p \in U^+(\mathfrak{g})^\mathfrak{g}$ there is $C_p \in \mathcal{W}(\mathfrak{g})^\mathfrak{g}$ and the transgression of p is defined by

$$\mathsf{t}_{\mathrm{Cl}(\mathfrak{g})}(p) = \pi_{\mathrm{Cl}(\mathfrak{g})}(C_p).$$

The transgression map

$$\mathsf{t}_{\mathrm{Cl}(\mathfrak{g})} \colon U^+(\mathfrak{g})^\mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})^\mathfrak{g}$$

satisfies

$$\operatorname{im} \mathsf{t}_{\mathrm{Cl}(\mathfrak{g})} = q(P_\wedge(\mathfrak{g})) = \operatorname{im} q \circ \mathsf{t}_{\wedge\mathfrak{g}}$$

Proposition (Kostant)

For $x \in \mathfrak{g}$ and $\phi \in P_\wedge(\mathfrak{g})$ we have that

$$\iota_x q(\phi) \in \operatorname{im} \alpha_\mathfrak{g} (\cong \mathrm{End}(V_\rho))$$

How we prove things? Noncommutative Weil aglebra.

If $E = \bigoplus E_i$ is a $\mathbb{Z}$-graded super space. For $n \in \mathbb{Z}$ define a $\mathbb{Z}$-grade super space $E[n]$ such that $E[n]_i = E_{i+n}$ (with shifted parities).

Define the noncommutative Weil algebra

$$\widetilde{W}(\mathfrak{g}) = T(\mathfrak{g}^*[-2] \oplus \mathfrak{g}^*[-1]).$$

$\widetilde{W}(\mathfrak{g})$ is G-differential algebra.

In the similar manner we define transgression maps

$$\widetilde{\mathrm{t}}_{\wedge\mathfrak{g}^*} : \text{"}\widetilde{W}^{+,0}(\mathfrak{g})^{\mathfrak{g}}\text{"} \to (\textstyle\bigwedge \mathfrak{g}^*)^{\mathfrak{g}}, \quad \widetilde{\mathrm{t}}_{\mathrm{Cl}(\mathfrak{g})} : \text{"}\widetilde{W}^{+,0}(\mathfrak{g})^{\mathfrak{g}}\text{"} \to \mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}.$$

$$
\begin{tikzcd}
 & W(\mathfrak{g})^{\mathfrak{g}} \arrow[dl, "\mathrm{d}_W"'] \arrow[drr, "\pi_{\wedge\mathfrak{g}^*}"] & & \\
S^+(\mathfrak{g})^{\mathfrak{g}} \arrow[rrr, "\mathsf{t}_{\wedge\mathfrak{g}^*}"] \arrow[dr, "\mathrm{sym}"] \arrow[dd, "q_W"] & & & (\bigwedge\mathfrak{g}^*)^{\mathfrak{g}} \arrow[dd, "q"] \\
 & \text{''}\widetilde{W}^{+,0}(\mathfrak{g})^{\mathfrak{g}}\text{''} \arrow[urr, "\widetilde{\mathsf{t}}_{\wedge\mathfrak{g}^*}"] \arrow[dl] \arrow[drr] & \widetilde{W}(\mathfrak{g})^{\mathfrak{g}} \arrow[l, "\mathrm{d}_{\widetilde{W}}"] \arrow[ur] \arrow[dr, "\widetilde{\mathsf{t}}_{\mathrm{Cl}(\mathfrak{g})}"'] & \\
U^+(\mathfrak{g})^{\mathfrak{g}} \arrow[rrr, "\mathsf{t}_{\mathrm{Cl}(\mathfrak{g})}"] & & & \mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}} \\
 & \mathcal{W}(\mathfrak{g})^{\mathfrak{g}} \arrow[ul, "\mathrm{d}_{\mathcal{W}}"] \arrow[urr, "\pi_{\mathrm{Cl}(\mathfrak{g})}"'] & &
\end{tikzcd}
$$

Plus explicit formulas for transgression maps.

The relative Weil algebra $W(\mathfrak{g}, K)$

Define the *relative Weil algebra*

$$W(\mathfrak{g}, K) = W(\mathfrak{g})_{K-\mathrm{bas}} = (S(\mathfrak{g}^*) \otimes \bigwedge \mathfrak{p}^*)^K$$

(which is just a differential algebra).

The restriction of the projection $\pi_{\wedge \mathfrak{g}^*}$:

$$\pi_{\wedge \mathfrak{p}^*} : W(\mathfrak{g}, K) \to (\bigwedge \mathfrak{p}^*)^K.$$

We have that $H(W(\mathfrak{g}, K)) \cong S(\mathfrak{g}^*)^K$.

Moreover, for $p \in S(\mathfrak{g}^*)^{\mathfrak{g}}$ we have that $[p] = 0$ iff $\mathrm{pr}_{\mathfrak{k}}(p) = 0$, where $\mathrm{pr}_{\mathfrak{k}} : S(\mathfrak{g}^*)^{\mathfrak{g}} \to S(\mathfrak{k}^*)^K$. Set

$$T(\mathfrak{g}, K) = \{p \in S(\mathfrak{g}^*)^{\mathfrak{g}} \mid \pi_{\mathfrak{k}}(p) = 0\}$$

For $p \in T(\mathfrak{g}, K)$ there is a $C_p^{\mathrm{rel}} \in W(\mathfrak{g}, K)$ such that $\mathrm{d}_W C_P^{\mathrm{rel}} = p$. (In general it is not the same as C_p in $W(\mathfrak{g})^{\mathfrak{g}}$!)

The *relative transgression map* $\mathsf{t}_{\wedge \mathfrak{p}^*} : T(\mathfrak{g}, K) \to (\bigwedge \mathfrak{p}^*)^K$ is defined by

$$\mathsf{t}_{\wedge \mathfrak{p}^*}(p) = \pi_{\wedge \mathfrak{p}^*}(C_p^{\mathrm{rel}})$$

Lemma

For $p \in S^{m+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ *such that* $\mathrm{pr}_{\mathfrak{k}}(p) = 0$ *we have that*

$$\mathsf{t}_{\wedge\mathfrak{p}^*}(p) = -4^m \operatorname{Res}^{\mathfrak{g}}_{\mathfrak{p}} \circ \mathsf{t}_{\wedge\mathfrak{g}^*}(p)$$

The $\mathbb{Z}$-graded subspace of $P_\wedge(\mathfrak{p}) = \operatorname{Res}^{\mathfrak{g}}_{\mathfrak{p}} P_\wedge(\mathfrak{g})$ is called the space of *primitive invariants* in $\bigwedge\mathfrak{p}^*$.
For our $(\mathfrak{g}, \mathfrak{k})$ we get that (Serre–Hochschild)

$$\dim P_\wedge(\mathfrak{p}) = \dim \mathfrak{a}.$$

Theorem (Calvert–Grizelj–K.–Panžić)

For the following pairs $(\mathfrak{g}, \mathfrak{k})$

$$(\mathfrak{sl}(n), \mathfrak{o}(n)), \qquad (\mathfrak{sl}(2n), \mathfrak{sp}(2n)), \qquad (\mathfrak{e}(6), \mathfrak{f}(4)).$$

The relative transgression map satisfies

$$\operatorname{im} \mathsf{t}_{\wedge\mathfrak{p}^*} = P_\wedge(\mathfrak{p}), \qquad \ker \mathsf{t}_{\wedge\mathfrak{p}^*} = S^+(\mathfrak{g}^*)^{\mathfrak{g}} \cdot T(\mathfrak{g}, K).$$

The transgression map in $\mathcal{W}(\mathfrak{g}, K)$

We define the *relative quantum Weil* algebra by

$$\mathcal{W}(\mathfrak{g}, K) = \mathcal{W}(\mathfrak{g})_{K-\mathrm{bas}} = (U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^K.$$

One can define the transgression map for $\mathcal{W}(\mathfrak{g}, K)$ in the similar manner

$$\mathsf{t}_{\mathrm{Cl}(\mathfrak{p})}(p) \colon U^+(\mathfrak{g})^{\mathfrak{g}} \to \mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}}.$$

Which allows us to prove

Proposition

For the following pairs $(\mathfrak{g}, \mathfrak{k})$

$$(\mathfrak{sl}(n), \mathfrak{o}(n)), \qquad (\mathfrak{sl}(2n), \mathfrak{sp}(2n)), \qquad (\mathfrak{e}(6), \mathfrak{f}(4)).$$

1. *Set* $P_{\mathrm{Cl}}(\mathfrak{p}) = q(P_\wedge(\mathfrak{p}))$, *then* $\mathrm{im}\,\mathsf{t}_{\mathrm{Cl}(\mathfrak{p})} = P_{\mathrm{Cl}}(\mathfrak{p})$.
2. *For* $x \in \mathfrak{p}$ *we have that*

$$\iota_x(\mathrm{im}\,\mathsf{t}_{\mathrm{Cl}(\mathfrak{p})}) = \iota_x(P_{\mathrm{Cl}}(\mathfrak{p})) \subset \mathrm{im}\,\alpha_{\mathfrak{p}}$$

III. Harish–Chandra projections

Harish–Chandra projections I

- Fix a triangular decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-.$$

It induces decomposition of $\mathrm{Cl}(\mathfrak{g})$:

$$\mathrm{Cl}(\mathfrak{g}) = \mathrm{Cl}(\mathfrak{h}) \oplus \big(\mathrm{Cl}(\mathfrak{n}^+)\,\mathrm{Cl}(\mathfrak{g}) + \mathrm{Cl}(\mathfrak{g})\,\mathrm{Cl}(\mathfrak{n}^-)\big)$$

- There is a filtered algebra isomorphism

$$\mathrm{hc}\colon\ \mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}} \to \mathrm{Cl}(\mathfrak{h}),$$

where the filtration on $\mathfrak{h}$ is induced by the principal $\mathfrak{sl}_2$-embedding $\mathfrak{sl}_2 \to \mathfrak{g}$ (proved by Joseph, Alekseev–Moreau).

Harish–Chandra projections II

Fix a choice $\Delta^+(\mathfrak{k}, \mathfrak{t})$ of positive $\mathfrak{t}$-roots in $\mathfrak{k}$.

Each $w \in W^1$ fixes the choice $\Delta^+_w(\mathfrak{g}, \mathfrak{t})$ of positive $\mathfrak{t}$-root in $\mathfrak{g}$. Consider the corresponding triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_w^+ \oplus \mathfrak{h} \oplus \mathfrak{n}_w^-.$$

Set

$$\mathfrak{p}_w^+ := \mathfrak{p} \cap \mathfrak{n}_w^+, \qquad \mathfrak{p}_w^- := \mathfrak{p} \cap \mathfrak{n}_w^-, \qquad \mathfrak{p} = \mathfrak{p}_w^+ \oplus \mathfrak{a} \oplus \mathfrak{p}_w^-.$$

We have a decomposition

$$\mathrm{Cl}(\mathfrak{p}) = \mathrm{Cl}(\mathfrak{a}) \oplus \big(\mathrm{Cl}(\mathfrak{p}_w^+)\,\mathrm{Cl}(\mathfrak{p}) + \mathrm{Cl}(\mathfrak{p})\,\mathrm{Cl}(\mathfrak{p}_w^+)\big)$$

Proposition (Calvert–Grizelj–K.–Pandžić)

For each $w \in W^1$ the corresponding Harish–Chandra projection

$$\mathrm{hc}_w \colon \ \mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}} \to \mathrm{Cl}(\mathfrak{a})$$

is a surjective algebra homomorphism.

IV. Sketch of the proof of the main result

Proof I

For the following pairs $(\mathfrak{g}, \mathfrak{k}, K)$

$$(\mathfrak{sl}(2n+1), \mathfrak{o}(2n+1), SO(2n+1)), \quad (\mathfrak{sl}(2n+1), \mathfrak{o}(2n), O(2n)),$$
$$(\mathfrak{sl}(2n), \mathfrak{sp}(2n), Sp(2n)), \quad (\mathfrak{e}(6), \mathfrak{f}(4), F(4)).$$

Lemma

For any $w \in W^1$ the restriction of all hc_w to $\mathrm{Cl}(\mathfrak{p})^K$ is an algebra isomorphism. Moreover, the restrictions of all hc_w to $\mathrm{Cl}(\mathfrak{p})^K$ coincide.

Lemma

The Harish–Chandra projection $\mathrm{hc}_w \colon \mathrm{Cl}(\mathfrak{p}) \to \mathrm{Cl}(\mathfrak{a})$ maps $\alpha_{\mathfrak{p}}(U(\mathfrak{k}))$ to $\mathbb{C}$.

Theorem

For each $w \in W^1$ the Harish Chandra projection $\mathrm{hc}_w : \mathrm{Cl}(\mathfrak{p}) \to \mathrm{Cl}(\mathfrak{a})$, takes primitives $\phi \in P_{\mathrm{Cl}}(\mathfrak{p})$ to linear elements of $\mathrm{Cl}(\mathfrak{a})$, i.e., to $q(\bigwedge^1 \mathfrak{a})$. Moreover, $\mathrm{hc}_w(P_{\mathrm{Cl}}(\mathfrak{p})) = \mathfrak{a}$.

Proof. Use $\iota_x(\mathrm{hc}_w(\phi)) = \mathrm{hc}_w(\iota_x \phi)$ for all $x \in \mathfrak{a}$.

Corollary

For $\phi, \psi \in P_{\mathrm{Cl}}(\mathfrak{p})$ *we have that* $[\phi, \psi] \in \mathbb{C}$*.*

Corollary

For homogeneous $\phi, \psi \in P_{\wedge}(\mathfrak{p})$ *we have that*

$$[q(\phi), q(\psi)] = 2(\iota_\phi \psi)_{[0]} = 2\tilde{B}(\phi, \psi).$$

- By the universal property the inclusion $P_{\mathrm{Cl}}(\mathfrak{p}) \hookrightarrow \mathrm{Cl}(\mathfrak{p})^K$ extends to

$$f\colon \mathrm{Cl}(P_{\mathrm{Cl}}(\mathfrak{p}), \tilde{B}) \hookrightarrow \mathrm{Cl}(\mathfrak{p})^K$$

- The composition $\mathrm{hc}_w \circ f$ is an algebra isomorphism.
- $\mathrm{hc}_w(f(P_{\mathrm{Cl}}(\mathfrak{p}))) = \mathfrak{a}$.
- Hence the form on $\tilde{B}$ on $P_{\mathrm{Cl}}(\mathfrak{p})$ is nondegenerate.

V. Kostant's Clifford algebra conjecture

Kostant's Clifford algebra conjecture for $\mathfrak{g}$

For $\mathrm{Cl}(\mathfrak{g})$, $\mathrm{hc}\colon \mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}} \to \mathrm{Cl}(\mathfrak{h})$.

$$\mathrm{hc}(q(P_{\wedge}(\mathfrak{g}))) = \mathfrak{h} \subset \mathrm{Cl}^{(1)}(\mathfrak{h})$$

Define a filtration on $\mathfrak{h}$ by

$$F^{(m)}\mathfrak{h} = \bigoplus_{i=0}^{m} \mathrm{hc}(q(P_{\wedge}^{i}(\mathfrak{g}))).$$

Let $\check{\mathfrak{g}}$ be the Lie algebra defined the dual root system and $(\check{e}, \check{h}, \check{f})$ be the principal $\mathfrak{sl}_2$-triple in $\check{\mathfrak{g}}$. Note $\check{\mathfrak{h}}^* \cong \mathfrak{h}$.

$$\mathcal{F}^{(m)}\mathfrak{h} = \{h \in \mathfrak{h} \mid (\mathrm{ad}^*_{\check{e}})^{m+1} h = 0\}.$$

Theorem (Joseph, Aleekseev–Moreau'2012)

$$\mathcal{F}^{(m)}\mathfrak{h} = F^{(m)}\mathfrak{h} \qquad \textit{for all } m.$$

Kostant's Clifford algebra conjecture for $(\mathfrak{g}, \mathfrak{k})$

For $\mathrm{Cl}(\mathfrak{p})$, $\mathrm{hc}_w \colon \mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}} \to \mathrm{Cl}(\mathfrak{a})$.

$$\mathrm{hc}_w(q(P_\wedge(\mathfrak{p}))) = \mathfrak{a} \subset \mathrm{Cl}^{(1)}(\mathfrak{a})$$

Define a filtration on $\mathfrak{a}$ by

$$F^{(m)}\mathfrak{a} = \bigoplus_{i=0}^{m} \mathrm{hc}_w(q(P^i_\wedge(\mathfrak{p})).$$

Let $\check{\mathfrak{g}}$ be the Lie algebra defined the dual root system and $(\check{e}, \check{h}, \check{f})$ be the principal $\mathfrak{sl}_2$-triple in $\check{\mathfrak{g}}$.
Note $\mathfrak{a} \subset \mathfrak{h} \cong \check{\mathfrak{h}}^*$.

$$\mathcal{F}^{(m)}\mathfrak{a} = \{h \in \mathfrak{a} \mid (\mathrm{ad}^*_{\check{e}})^{m+1} h = 0\}.$$

Theorem (Calvert–Grizelj–K.–Pandžić)

$$\mathcal{F}^{(m)}\mathfrak{a} = F^{(m)}\mathfrak{a} \qquad \textit{for all } m.$$

What is next?

• (noncommutative) equivariant cohomology (Alekseev–Meinrenken); different Cartan's models.

• (quasi-)Poisson geometry and (group-valued) moment maps.

• Cohomologies of the moduli space of flat connections on G-bundle over a Riemannian surface.

• Schubert calculus and cohomologies of $G/K)$ and their q-deformations.

• Higher gauge theories and L_∞-formalisim.

• (algebraic) Dirac operators and their generalisations.
– Dirac induction?
– for quantum groups (K.–Pandžić)

$$(\textstyle\bigwedge_q \mathfrak{sl}_2)^{U_q(\mathfrak{sl}_2)} = \textstyle\bigwedge_q P_{\wedge_q}(\mathfrak{sl}_2), \quad \mathrm{Cl}_q(\mathfrak{sl}_2)^{U_q(\mathfrak{sl}_2)} = \mathrm{Cl}(P_{\mathrm{Cl}_q}(\mathfrak{sl}_2), B_q)$$

Thank you!